The logic of the prime number distribution

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Abstract

There are clear rules which define the distribution and the existence of prime numbers. But not the prime numbers define these rules ! The non-prime numbers between the prime numbers are the defining elements of these rules !

In principle there are two basic number sequences which play a major role in the prime number distribution. The first Number Sequence (SQ1) contains all prime numbers of the form 6n+5 and the second Number Sequence (SQ2) contains all prime numbers of the form 6n+1.
All existing prime numbers are contained in these two number sequences except of the prime numbers 2 and 3.

Another key role in the distribution of the prime number plays the number 5 and its periodic occurrence in the two number sequences SQ1 & SQ2. All non-prime numbers in SQ1 & SQ2 are caused by recurrences of these two number sequences with increasing wave-lengths in themselves, in a similar fashion as "Overtones" (harmonics) or "Undertones" derive from a fundamental frequency.

On the contrary prime numbers represent "silent spots" in these two basic Number Sequences SQ1 & SQ2 where there is no interference caused by these recurring number sequences.

The distribution of the non-prime numbers and prime numbers can be described in a graphical way with a simple " Wave Model " (or Interference Model) → see Table 2.

1 To the distribution of non-prime numbers

To find a logic in the prime number distribution it seems to be advisable to have a proper look to the distribution of the non-prime numbers first.

Non-prime numbers are numbers which are products of prime numbers.

Therefore it is only logical, that the distribution of non-prime numbers needs the same attention in the search for a distribution law for prime numbers, as the distribution of prime numbers itself.

The trigger for my increased interest in the distribution of the non-prime numbers was my work on another study which I carried out in regards to the prime number distribution.
(→ Titel of this study : " The Distribution of Prime Numbers on the Square Root Spiral ")

In this previous study I realized an important fact :

I noted a clear defined logic in the periodic occurrence of the prime factors which form the non-prime numbers of special prime number sequences.

Another impessive fact which I noted in this study, was the pair of prime number sequences which runs along the square root spiral. In these two prime number sequences the distance from on prime number to the next is always 6. And the two prime number sequences are shifted to each other by a distance of 2. Of course these two sequences are based on the well known fact, that if a prime number is divided by 6 the rest of +1 or –1 remains. Or more precise : every prime number is either of the form 6n + 1 or 6n + 5.

I named these two "Prime Number Sequences" Sequence 1 and Sequence 2 (SQ1 & SQ2)

Table 1 shows these two sequences in the centre of the table.

Here all prime numbers in Sequence 1 & 2 are marked in yellow and all non-prime numbers are marked in green !

Sequence 1 starts with number 5 and Sequence 2 starts with number 1. The next numbers in each Sequence are always the previous number plus 6. From this rule the following sequences arise :

Sequence 1 (SQ1)	:	5, 11, 17, 23, 29, **35**, 41, 47,….	and
Sequence 2 (SQ2)	:	1, 7, 13, 19, **25**, 31, 37, 43,….	

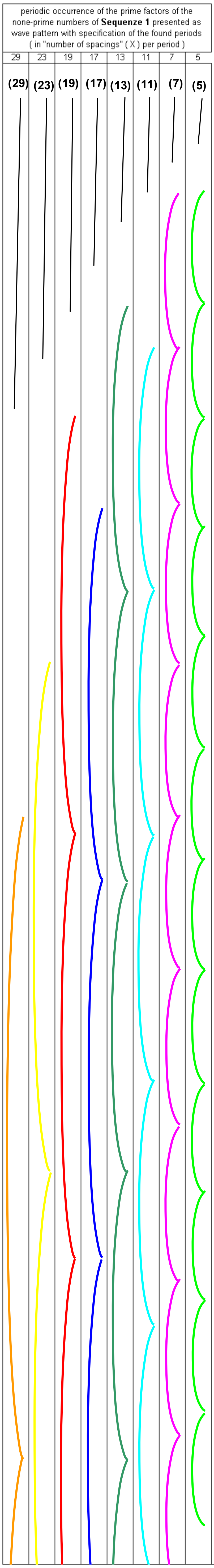
It is easy to see, that all existing prime numbers seem to be contained in these two sequences, except the two prime numbers 2 and 3 ! Further it is notable that in both sequences there are no numbers which are divisible by 2 or 3.

Sequence 1 & 2 not only contain prime numbers, but also non-prime numbers which consist of certain prime factors, e.g. the numbers **25** and **35**. These non-prime numbers are marked in green. Beside these non-prime numbers, in the column "Prime Factors of the Non-Prime Numbers" their prime factors are shown. (e.g. the prime factors of 35 are 5 and 7 → 5 x 7 = 35).

Further there are additional colums available beside Sequence 1 and Sequence 2, which I used to explain the logical principle of the distribution of the prime factors in the non-prime numbers in these two sequences.

Now I want to invite the attentive reader to have a closer look at Table1 to get a "feeling" of the logic which is determining the distribution of the non-prime numbers !

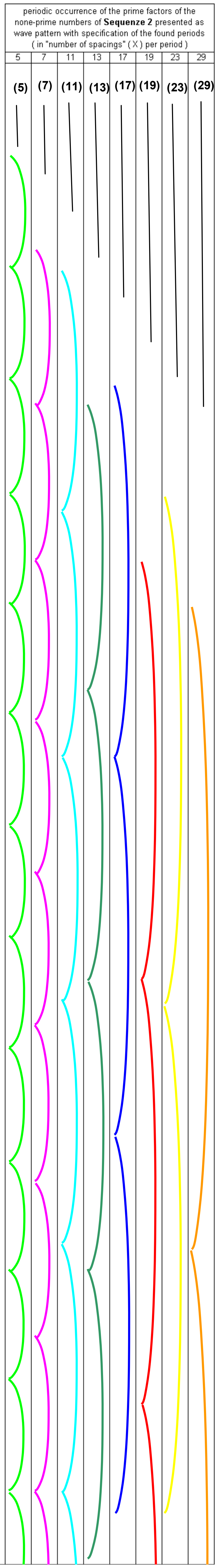
TABLE 1 : Shows the periodic occurrence of the prime-factors which form the "non-prime numbers" in **Sequence 1 & 2**



CORE PERIODICITY of Sequenze 1 & Sequenze 2

Prime Number Sequenze 1	prime factors of the Non-prime numbers	29	23	19	17	13	11	7	5
5									
11									
17									
23									
29									
35	5 x 7							7	5
41									
47									
53									
59									
65	5 x 13					13			5
71									
77	7 x 11						11	7	
83									
89									
95	5 x 19			19					5
101									
107									
113									
119	7 x 17				17			7	
125	5 x 5 x 5								5
131									
137									
143	11 x 13					13	11		
149									
155	5 x 31								5
161	7 x 23		23					7	
167									
173									
179									
185	5 x 37								5
191									
197									
203	7 x 29	29						7	
209	11 x 19			19			11		
215	5 x 43								5
221	13 x 17				17	13			
227									
233									
239									
245	5 x 7 x 7							7	5
251									
257									
263									
269									
275	5 x 5 x 11						11		5
281									
287	7 x 41							7	
293									
299	13 x 23		23			13			
305	5 x 61								5
311									
317									
323	17 x 19			19	17				
329	7 x 47							7	
335	5 x 67								5
341	11 x 31						11		
347									
353									
359									
365	5 x 73								5
371	7 x 53							7	
377	13 x 29	29				13			
383									
389									
395	5 x 79								5
401									

Prime Number Sequenze 2	prime factors of the Non-prime numbers	5	7	11	13	17	19	23	29
1									
7									
13									
19									
25	5 x 5	5							
31									
37									
43									
49	7 x 7		7						
55	5 x 11	5		11					
61									
67									
73									
79									
85	5 x 17	5				17			
91	7 x 13		7		13				
97									
103									
109									
115	5 x 23	5						23	
121	11 x 11			11					
127									
133	7 x 19		7				19		
139									
145	5 x 29	5							29
151									
157									
163									
169	13 x 13				13				
175	5 x 5 x 7	5	7						
181									
187	11 x 17			11		17			
193									
199									
205	5 x 41	5							
211									
217	7 x 31		7						
223									
229									
235	5 x 47	5							
241									
247	13 x 19				13		19		
253	11 x 23			11				23	
259	7 x 37		7						
265	5 x 53	5							
271									
277									
283									
289	17 x 17					17			
295	5 x 59	5							
301	7 x 43		7						
307									
313									
319	11 x 29			11					29
325	5 x 5 x 13	5			13				
331									
337									
343	7 x 7 x 7		7						
349									
355	5 x 71	5							
361	19 x 19						19		
367									
373									
379									
385	5 x 7 x 11	5	7	11					
391	17 x 23					17		23	
397									
403	13 x 31				13				



2 Analysis of the occurrence of prime factors in the non-prime numbers of SQ1 & SQ2

Looking at Table 1 the following facts are pretty obvious and easy to see ;

- **The prime factors of all non-prime numbers occur periodic in both sequences.**

  e.g. note the periodic occurrence of the prime factors 5 and 7 → shown by the green and pink wavy lines beside Sequence 1 and Sequence 2.
  The periodic occurrence of the prime factors 5, 7, 11, 13, 17, 23 and 29 is also presented in tabular form and as a separate wavy line on the righthand- and lefthand-side of Table 1.

- **The occurrence period of each prime factor is linked (equal) to its value**

  e.g. prime factor 5 occurs in every fifth number of Sequence 1 & 2, prime factor 7 occurs in every seventh number of these sequences and so on……
  This rule seems to be valid for all prime factors of all non-prime numbers !

These two facts are already pretty interesting ! But this is only the beginning !! There is more and more a well organized structure emerging, which precisely defines the position of every prime factor of the non-prime numbers.
The next facts are not so easy to see at first sight, but with a bit concentration everyone can understand them pretty quick.

- **New prime-factors in the non-prime numbers only occur together with prime factor 5 (or 7)**

  In the column "prime factors of the non-prime numbers" I marked all prime factors which occurred for the first time in a pink color. In the column where I presented the prime factors of the non-prime numbers in tabular form, I additional marked the prime factors 5 and 7 in a green color and the prime factors 11, 13, 17, 23 and 29 in an orange color, when they occurred for the first time. I followed the non-prime numbers in Sequenze 1 & 2 for quite a while. And the rule that new prime factors in Sequence 1 or 2 only emerge together with the prime factors 5 or 7 is always valid ! If we consider Sequence 1 and 2 simultaneously then it even applies that new prime factors **at first** only occur together with the prime factor **5** !!!

- **The prime factors which occur together with the prime factors 5 or 7, form sequences which are equivalent to Sequence 1 & 2**

  It turns out that the prime factors of the non-prime numbers, which occur together with the prime factors **5** or **7** (→ marked in pink in Table 1), form number sequences which are equivalent to Sequence 1 & 2 !! This is relatively easy to see with the help of the green and pink wavy lines beside Sequence 1 & 2.
  For example : by following the green and pink wavy lines beside **Sequence 1** it is noticable that the prime factors of the non-prime numbers, which occur together with prime factor **5** or **7**, form the following two "Sub-Sequences" :

  Sequence of the prime factors which occur together with prime factor **5** in Sequence **1** :
  7, 13, 19, **25**, 31, 37, 43, **49**, **55** , 61, 67, 73, ….. (→ equivalent to Sequence **2**)

  Sequence of the prime factors which occur together with prime factor **7** in Sequence **1** :
  5, 11, 17, 23, 29, **35**, 41, 47, 53, 59, **65**, 71, **77**, …….. (→ equivalent to Sequence **1**)

It is obvious that these "Sub-Sequences" of the other prime factors, which occur together with the prime factor **5** or **7** (shown above) are equivalent to Sequence 1 & 2 (SQ1 & SQ2) in Table 1.

3 “Wave Model” for the distribution of the Non-Prime Numbers & Prime Numbers

The lefthand side of **Table 2** shows the periodic occurrence of the numbers divisible by 5, 7 or 11. The periodic distribution of these numbers in Sequence 1 & 2 can be grasped instantly, with the help of the wavy lines which indicate the positions of these numbers.

This graphical representation of the periodic occurrence of the prime factors 5, 7 and 11 is used as a base for a relatively simple “wave model”, to explain the distribution of Non-Prime Numbers and Prime Numbers in Sequence 1 & 2.

The base of this “Wave Model” is the fact, that Sequence 1 & 2 recurs in itself with increasing wave-lengths, in a similar way as “Undertones” derive from a defined fundamental frequency **f**.

“Undertones” are the inversion of “Overtones” which are well known by every musician, because they occur in nearly every musical instrument.

Overtones (harmonics) are integer multiples of a fundamental frequency **f**.

On the contrary an “Undertone” theoretically results from inverting the principle of creation of an “Overtone”.

The misconception that the undertone series is purely theoretical rests on the fact, that it does not sound simultaneously with its fundamental tone, as the overtone series does. It is, rather, opposite in every way.

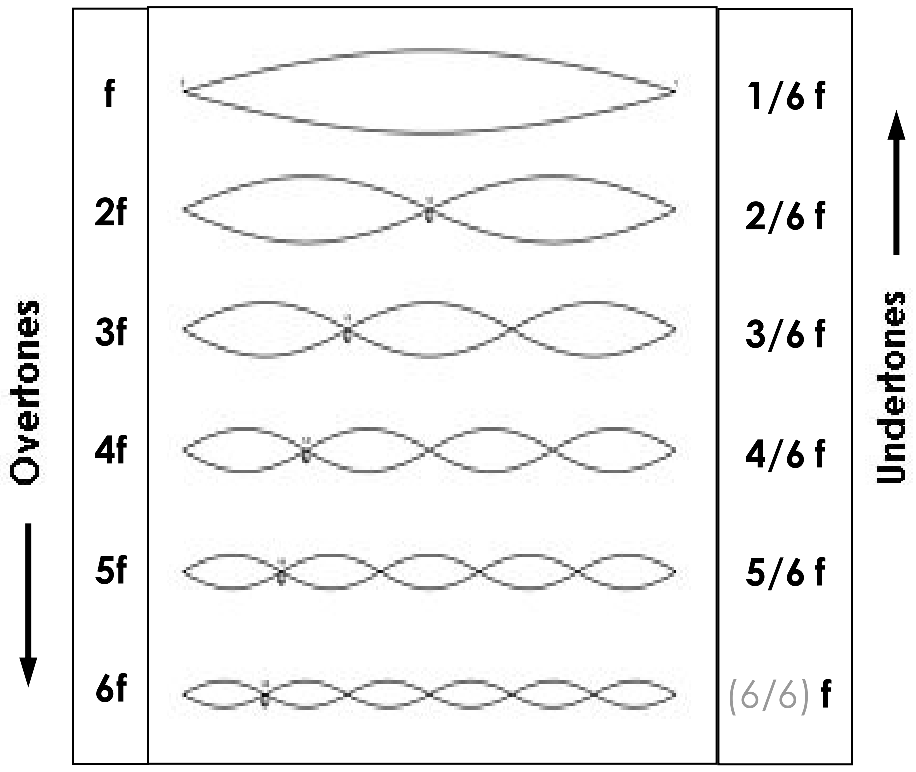


FIG. 1 : **Overtones** (or “ harmonics”) and **Undertones**, deriving from a fundamental frequency **f**

We produce the overtone series in two ways : either by overblowing a wind instrument, or by dividing a monochord string. If we lightly damp the monochord string at the halfway point, then at 1/3, then 1/4, 1/5, etc., we produce the overtone series *2f*, *3f*, *4f*, 5f,... etc. , which includes the major triad.

If we simply do the opposite, we produce the **Undertone Series**, i.e. by multiplying.
A short bit of string will give a high note. If we can maintain the same tension while plucking twice, 3 times, 4 times that length, etc. the undertone series 1/2f, 1/3f, 1/4f,... etc. will unfold containing the minor triad. Of course there are also undertones possible in between, for example at 5/6f or 2/3f (4/6f) as the above Diagram (FIG. 1) shows !

In **FIG.1** it is easy to see that every given “base frequency” only allows a finite number of “undertone frequencies” ! And the number of possible undertone frequencies is definite !

Coming back now to our “Wave-Model” : Here Sequence 1 & 2 represents the “Base Oscillation” with the frequency f from which “Undertone Oscillations” with the frequencies (1/x)f derive. And these Undertone Oscillations define the distribution of the non-prime numbers in Sequence 1 & 2.

The first Undertone Oscillation 5 has a frequency of 1/5 of the Base Oscillation. The next Undertone Oscillation has a frequency of 1/7 and the next one 1/11 , then 1/13 and so one…
These means that the “wave lengths” of the Undertone Oscillations, which derive from the Base Oscillation, are 5 times, 7 times, 11 times , 13 times , ... etc. longer than the wave length of the Base Oscillation. It is obvious that the sequence of frequencies or wave lengths of the Undertone Oscillations again represents the Number Sequences 1 & 2 (SQ1 & SQ2) !!

Here now some important properties of the “Wave Model”, which define the distribution of the Non-Prime Numbers and Prime Numbers in Sequence 1 & 2 : (→ see Table 2)

4 Properties of the "Wave Model" shown in Table 2 :

- **Non-prime-numbers are caused by "Undertone Oscillations", which derive from a Base-Oscillation (with the fundamental frequency f) which is defined by the Number Sequences 1 & 2 (SQ1 & SQ2).**

- **These Undertone Oscillations have frequencies and wave-lengths which are defined by the numbers contained in Sequence 1 & 2.**

  __For example__ : The first Undertone Oscillations have the frequencies 1/**5**f, 1/**7**f, 1/**11**f, 1/**13**f, 1/**17**f, 1/**19**f, 1/**23**f ...etc. in comparison with the Base Oscillation which has the fundamental frequency **f** . It is easy to see, that the occurring "Undertone Frequencies" are defined by the numbers contained in Sequence 1 & 2

- **Every peak of an Undertone-Oscillation corresponds to a non-prime number in Sequence 1 & 2**

- **On the contrary "Prime Numbers" represent places in Sequence 1 & 2 which do not correspond with any peak of an Undertone-Oscillation.**

  Or to say it in other words : prime numbers represent "silent spots" in the two basic Number-Sequences SQ1 & SQ2 where there is no interference caused by the Undertone Oscillations

- **In every Undertone Oscillation "further Undertone Oscillations" occur, which again are defined by the numbers contained in Sequence 1 & 2.**
  However these "further Undertone Oscillations" are not required to explain the existence of the non prime numbers in Sequence 1 & 2, because the non prime numbers in these sequences are already explained by the undertone oscillations which directly derive from Sequence 1 & 2.

  (→ "further Undertone Oscillations" are marked by red circles on the corresponding peaks of the Undertone Oscillations. And the prime factor products of the numbers which belong to these peaks are shown in red and pink boxes)

  __**Example**__ : The numbers 125, 175, 275 and 325 in the Undertone Oscillation **5** (=1/5f), represent the prime factor products **5**x5x**5**, **5**x5x**7**, **5**x5x**11** and **5**x5x**13.** It is easy to see that these prime factor products form another Undertone Oscillation **5** in the Undertone Oscillation **5** !!

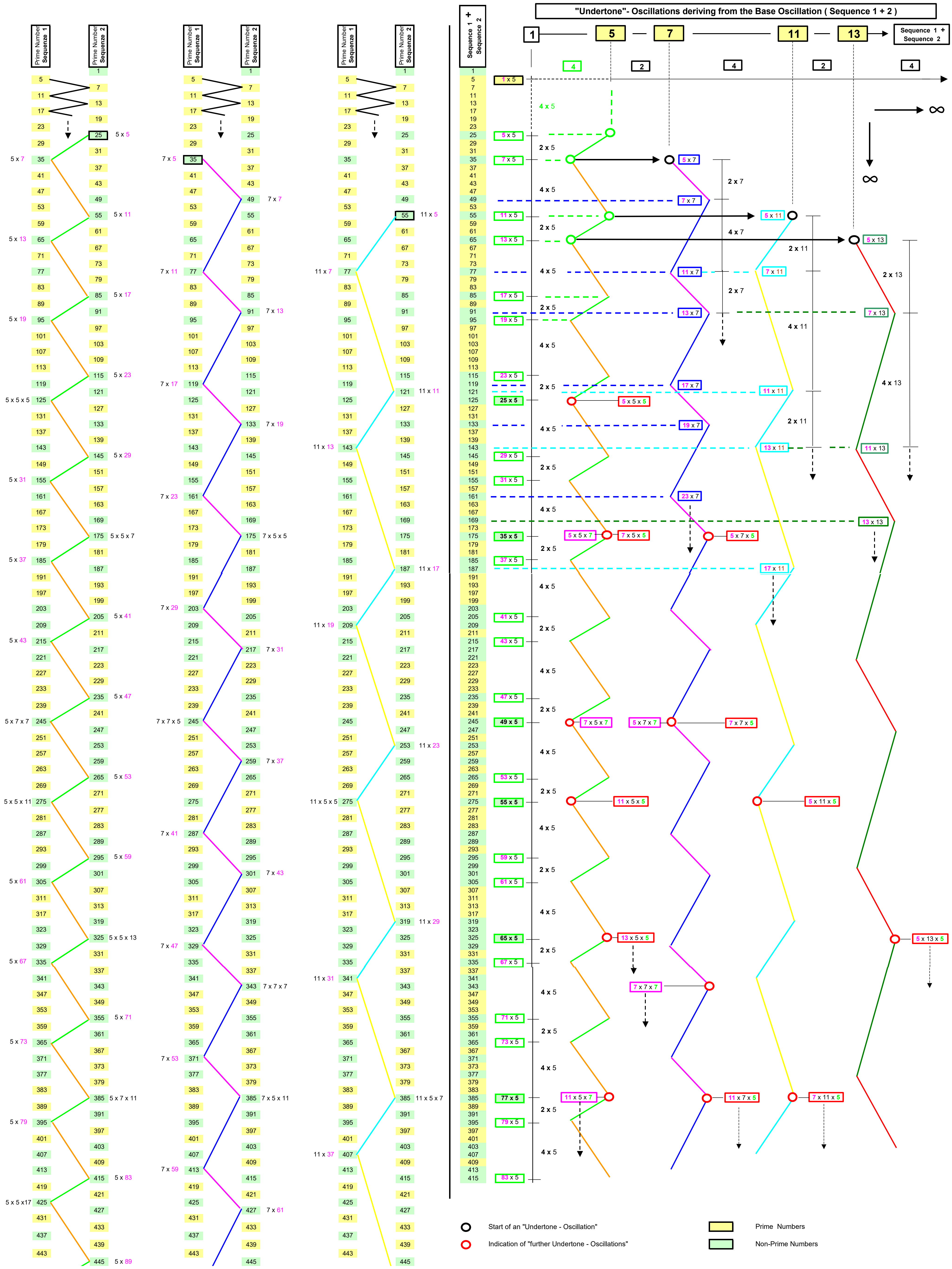
- **On every peak of the Undertone Oscillation 5 (=1/5 f) another Undertone Oscillation starts.**

  The green circles on the first few peaks of the Undertone Oscillation **5** mark the starting points of the next 3 Undertone Oscillations **7, 11** and **13** (= 1/**7**f, 1/**11**f and 1/**13**f). More of such Undertone Oscillations will start on every peak of the Undertone Oscillation **5** ad infinitum. Note that Undertone Oscillations which are defined by non-prime numbers (e.g. 1/**25**f or 1/**35**f etc.) are not required to explain the non-prime numbers in Sequence 1 & 2 !!

- **The sequence of the companion (prime)- factors in every Undertone-Oscillation is always 5,** 7, **11,** 13, **17,** 19, **23** ...

  In every Undertone Oscillation it applies, that the sequence of the companion (prime) factors, which form the numbers in this Undertone Oscillation, always starts with **5** and alternately increases by 2 and 4. These sequence of (prime) factors is equal to Sequence 1 + 2 except that the number 1 is missing !

TABLE 2 : "Wave Model" for the distribution of the Non-Prime Numbers & Prime Numbers in Sequence 1 & 2



5 General description of the "Wave Model" and the prime number distribution in mathematical terms

Definition of **Sequence 1 & 2** (Base oscillation with frequency **f**) in mathematical terms :

SQ1 (Sequence 1) : $a_n = 5 + 6n$ for example $a_0 = 5$; $a_1 = 11$; $a_2 = 17$ etc.

SQ2 (Sequence 2) : $b_n = 1 + 6n$ for example $b_0 = 1$; $b_1 = 7$; $b_2 = 13$ etc.

with $n \in N = \{0, 1, 2, 3, 4, \ldots\}$

Description of the "**Undertone Oscillation 5**" (= 1/**5** f) :

→ undertone oscillation **5** is split into two number sequences **U-5**$_1$ and **U-5**$_2$:

U-5$_1$: $a(5)_n = 5(\,5 + 6n\,)$ for example $a_0 = 25$; $a_1 = 55$; $a_2 = 85$ etc.

U-5$_2$: $b(5)_n = 5(\,1 + 6n\,)$ for example $b_1 = 35$; $b_2 = 65$; $b_3 = 95$ etc.

with $n \in N = \{0, 1, 2, 3, 4, \ldots\}$ for **U-5**$_1$ and with $n \in N^* = N\backslash\{0\} = \{1, 2, 3, 4, \ldots\}$ for **U-5**$_2$

General description of all "**Undertone Oscillations X**" (= 1/**X** f) :

→ every undertone oscillation is split into two number sequences **U-(x)**$_1$ and **U-(x)**$_2$:

U-(x)$_1$: $a(x)_n = x(\,5 + 6n\,)$ with $n \in N = \{0, 1, 2, 3, 4, \ldots\}$

U-(x)$_2$: $b(x)_n = x(\,1 + 6n\,)$ with $n \in N^* = N\backslash\{0\} = \{1, 2, 3, 4, \ldots\}$

and with $X \in (\,SQ1 \cup SQ2\,)\backslash\{1\} = \{5, 7, 11, 13, 17, 19, 23, 25\ldots\}$ for both sequences $a(x)_n$ & $b(x)_n$

According to the above described definitions the set of prime numbers (PN) can be defined as follows :

$$PN^* = (\,SQ1 \cap SQ2\,) \backslash (\,U\text{-}(x)_1 \cap U\text{-}(x)_2\,)$$

$$PN = \{2, 3\} \cap (\,SQ1 \cap SQ2\,) \backslash (\,U\text{-}(x)_1 \cap U\text{-}(x)_2\,)$$

for **PN*** and **PN** the following definition applies :

PN = { 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, …} ; **set of prime numbers**

and **PN*** = { 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, …} = **PN** \ { 2, 3 }

further the following definitions applies :

$PN^* \subset (\,SQ1 \cap SQ2\,)$ or $PN^* \subset (\,a_n \cap b_n\,)$

$PN^* \not\subset (\,U\text{-}(x)_1 \cap U\text{-}(x)_2\,)$ or $PN^* \not\subset (\,a(x)_n \cap b(x)_n\,)$ with $x \in (\,SQ1 \cup SQ2\,) \backslash \{1\}$

$NPN^* = (\,U\text{-}(x)_1 \cap U\text{-}(x)_2\,)$ or $NPN^* = (a(x)_n \cap b(x)_n)$; **NPN*** = non-prime-numbers not divisible by 2 or 3

6 Conclusion and final comment

My "Wave Model" shown in Table 2 is based on the fact, that Number Sequence SQ1 & SQ2 recurs in itself over and over again with increasing wave-lengths, in a similar way as "Undertones" or "Overtones" derive from a defined fundamental frequency **f** (see explanation in chapter 3).

This recurrence of Number Sequence SQ1 & SQ2 with increasing wave-lengths in itself, can be considered as the principle of creation of the non-prime numbers in SQ1 & SQ2. This fact is easy to see on the righthand side of Table 2, where Number Sequence SQ1 & SQ2 recurs in an ordered manner in the prime factors of its non-prime numbers.

Non-prime numbers are created on all places in the Number Sequences SQ1 & SQ2 where there is interference caused through the recurrences of these Number Sequences. On the other hand Prime Numbers represent "silent spots" in Number Sequence SQ1 & SQ2 where there is no such interference caused through the recurrences of this Number Sequences.

The distribution of prime numbers directly results from the distribution of the non-prime numbers, which can be described by a physical principle which is known from acoustics as "The harmonics of a wave ". By definition a harmonic (overtone) is an exact integer multiple of a fundamental frequency. For example, if the fundamental frequency is *f*, the harmonics (overtones) have the frequencies 2*f*, 3*f*, 4*f*, 5f, 6f, … etc.
Many oscillators, including the human voice, a bowed violin string, or a Cepheid variable star, are more or less periodic, and thus can be decomposed into harmonics.

Therefore it seems that the distribution of Prime Numbers is very much related to physics !

And all fields of physics which deal with waves and oscillations like acoustics, optical physics, quantum mechanics and quantum field theory might be good starting points to look for a final theory and the exact reason for the distribution and existence of Prime Numbers !

Professionals who work in the field of prime number theory should also have a look through my study, which refers to the distribution of prime numbers on the square root spiral , because this study might be helpful for a more profound insights in the distribution of prime numbers !
(Title of this study : "The Distribution of Prime Numbers on the Square Root Spiral")

Especially interesting should here be chapter 6, which deals with the periodic occurrence of prime factors in the non-prime numbers of the analysed polynomials and the comparison of the Square Root Spiral, Ulam Spiral and Number Spiral in chapter 10. The periodicities of the prime factors described in these two chapters are another confirmation, that there are clear rules which define the distribution of prime numbers.

Another interesting fact, which might be worth mentioning, is the constant occurrence of sub-sequences of 5 numbers in the prime number sequences described in the chapters 4.3 and 4.4 in the mentioned study. This indicates, that there is a kind of "basic-oscillation" existing in the Square Root Spiral, which always covers 5 windings of the Square Root Spiral per oscillation, and which interacts with all analysed Prime Number Spiral Graphs.

Finally I want to make a suggestion for the practical use of my Wave Model described in Table 2 :

Because my "Wave Model" described in Table 2, represents an easy way to explain the distribution of non-prime numbers and prime numbers in a visual way it shouldn't be very difficult to develop a computer program out of it, which can automatically extend this wave model with a high speed. The main task of this program would be the recognition (identification) of "peaks" in the Undertone Oscillations and the marking of their corresponding positions (or interferences) with Sequence 1 and Sequence 2 as "Non-Prime Numbers".
All remaining unmarked positions on Sequence 1 and 2 would then represent Prime Numbers.

This as encouragement to make use of my wave model for the discovery and registration of large Prime Numbers !

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